

Exercise 1:

You have the following available moves:

(A) $\bullet \circ \rightarrow (\text{Nothing})$

(B) $\bullet \bullet \rightarrow (\text{Nothing})$

(C) $\circ \bullet \rightarrow (\text{Nothing})$

(D) $\circ \circ \rightarrow (\text{Nothing})$

Try to reduce each of the following bead sequences as much as possible. Is it always possible to remove all of them? Why/why not?

a) $\bullet \circ \circ \bullet \bullet \circ \bullet \circ \circ \circ \bullet \circ$

b) $\bullet \circ \circ \bullet \circ \bullet \circ$

c) $\bullet \circ \circ \bullet \bullet \circ \bullet \circ \circ \circ \bullet$

Exercise 2:

You have the following available moves:

(A) $\bullet \circ \rightarrow \circ \bullet \circ$

(B) $\bullet \bullet \rightarrow (\text{Nothing})$

(C) $\circ \bullet \rightarrow \circ \circ \bullet$

(D) $\circ \rightarrow (\text{Nothing})$

If you start with $\circ \circ \circ \bullet \circ$, can you get rid of all of the beads? Hint: what happens to the number of painted beads with each move?

Challenge: which bead sequences can you get with this starting sequence and operations?

Problem 1

Start with $\bullet\bullet$. The allowed moves are:

(A) $\bullet\circ \rightarrow \circ\bullet$

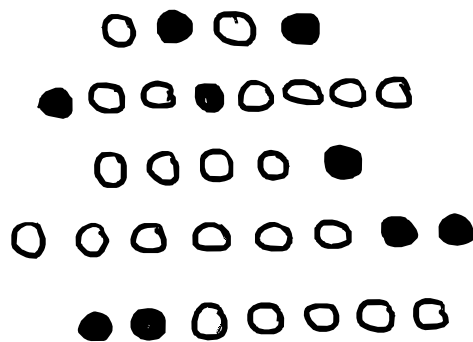
(B) $\circ\bullet \rightarrow \bullet\circ$

(C) $\bullet \rightarrow \circ\bullet\circ$

(D) $\circ\circ \rightarrow (\text{Nothing})$

a) Try to reach the following bead sequences:

(some are not possible, so if you get stuck move on for now)



b) Look at the impossible-looking sequences and try to understand why they are impossible.

c) What types of sequences can we make with these rules?

Hint: how many painted beads can we have? why?
how many unpainted? why?
is any order possible? why?

d) Does your answer to c) change if we remove moves (A) and (B)?

Hint: try to make moves (A) and (B) from (C) and (D)

Problem 2

Start with the bead sequence:



The allowed moves are:

(A) $\bigcirc \bullet \rightarrow \bigcirc \bullet \bullet$

(B) $\bigcirc \bigcirc \rightarrow \text{(Nothing)}$

(C) $\bullet \bullet \bullet \rightarrow \text{(Nothing)}$

What is the shortest bead sequence you can make using these rules?

Problem 3

Start with a single bead $\bullet$. The allowed moves are:

(A) You can double the sequence
(for example, $\bullet \bigcirc \bigcirc$ becomes $\bullet \bigcirc \bigcirc \bullet \bigcirc \bigcirc$)

(B) If a sequence ends in $\bullet$ you can add $\bigcirc$ in the end

(C) $\bigcirc \bigcirc \rightarrow \text{(Nothing)}$

(D) $\bullet \bullet \bullet \rightarrow \bigcirc$

Can you reach the sequence $\bigcirc$ with these rules?

Hints for problem 3:

- Try to make sequences with only painted beads. What are the options for the number of painted beads?

• Turn the sequence  into . What sequences with 4 painted beads and 2 unpainted beads can be done?

- More generally, what are the possible numbers of painted vs unpainted beads? For example, is 5 ● 2 ○ possible? Is 6 ● 2 ○?
- What are all the sequences we can make?

Problem 4

(Math Olympiad Problem!)

Start with the bead sequence:

Notice how similar
it looks to 2,
but it's harder!



The allowed moves are:

- (A)  $\rightarrow$ 
- (B)  $\rightarrow$ (Nothing)
- (C)  $\rightarrow$ (Nothing)

What is the shortest bead sequence you can make using these rules?

Hint (backwards because it solves most of the problems, do not read if you don't want spoilers):

Add the number of black beads until you reach a white, then subtract until you reach another white, then add, then subtract... The number you get never changes remainder when divided by 3. For example, you add 2, then subtract 6 then add 1 then subtract 1 $\rightarrow 2$

Problem 5

Available moves:

(A) ● ● → ○

(C) ○ ● → (Nothing)

(B) ○ ○ → ●

(D) ● ○ → (Nothing)

What is the smallest bead sequence you can reach if you start with:

a) ● ○ ● ○ ○ ○ ●

b) ● ● ● ● ● ●

c) ● ● ● ● ●

d) ● ○ ○ ● ● ○ ● ○ ○ ○

Problem 6

Start with

● ○ ● ○ ⊕ ⊕ ● ● ●

third bead color!

What is the smallest bead sequence you can reach with the moves:

a) (A) ● ○ → ⊕

(B) ⊕ ⊕ → ●

(C) ● ● ● → (Nothing)

b) (A) ● ○ → ⊕

(B) ○ ⊕ → ●

(C) ⊕ ● → ○